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PLASTIC FLOW OF AXIALLY-SYMMETRIC NOTCHED BARS PULLED IN TENSION

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Abstract—Theoretical and experimental results are presented for plastic flow of axially symmetric notched bars. The slip-line fields have been calculated numerically using the Shield theory of axially symmetric plastic flow. The load factors obtained from the theoretical solutions differ only slightly from those for plane strain conditions. Two sets of specimens were investigated in order to obtain the actual load factors for V-shaped and rounded notches. The agreement between the theory and experimental results is satisfactory. In the last series of experiments the influence of the diameter of the bar outside the notch was investigated and later compared with the theoretically estimated value of that diameter.

INTRODUCTION

THE PROBLEM of the stress distribution and mode of deformation of notched bars undergoing tension is rather well elaborated for plane strain or plane stress conditions only. For axially symmetric notched bars the solution based on the von Mises yield criterion and the associated flow rule is still unavailable, since the system of equations is not hyperbolic and therefore the method of characteristics cannot be used.

In the present paper a solution is presented for the Tresca yield criterion with the associated flow rule, along with the Haar–Karman hypothesis. The latter requires that in the plastic state the circumferential stress be equal to one of the principal stresses in the meridional plane.

Shield [1] has shown that such assumptions lead to a hyperbolic system of equations, and presented a complete solution of indentation of a semi-infinite plastic body by a flat axially symmetric punch. McClintock [2] has pointed out, that by simply changing the signs and indexes of stresses and velocities, the stress and velocity field can be obtained, corresponding to the plastic state of axially symmetric bar with a slit-shaped notch.

This paper contains a theoretical and experimental analysis of the yield point load for axially symmetric bars with V-shaped and various rounded notches. The theoretical part of the analysis includes numerical calculations of the yield load for V-notches of the total angle $2\gamma = 60°$ and $120°$, for a circular notch, and for a notch with a short cylindrical part bounded by two rounded parts. For each of these solutions an appropriate velocity field and the initial plastic deformation of the free surface of the notch may be found. All stress fields may be extended into the rigid parts of the bar outside the notch in the manner presented by Shield for the punch indentation problem. Thus solutions are complete, provided the diameter of the bar outside the notch is so large that the boundary of the extended slip line field lies entirely within the contour of the bar in meridional cross section. Since calculations of the extended stress field are very laborious, simple evaluation of the bar diameter for practical purposes is proposed.

Experiments were performed to determine the yield point and ultimate load of the bars with various notch parameters and to compare their results with the theoretical values. The influence of the bar diameter outside the notch was also investigated.

BASIC EQUATIONS

A detailed analysis of the axially symmetric flow of a rigid-plastic material obeying the Tresca yield criterion and the associated flow rule is given by Shield [1]. To make this paper sufficiently self-contained a number of necessary equations will be given below.

The stresses at any point of the plastic region of the axially symmetric notched bar are represented in the principal stress space $\sigma_1\sigma_2\sigma_3$ by points lying on that edge of the Tresca hexagonal prism for which

$$\sigma_2 = \sigma_3, \tag{1}$$

and

$$\sigma_1 - \sigma_2 = 2k, \tag{2}$$

where σ_1 and $\sigma_2 (\sigma_1 > \sigma_2)$ are the principal stresses in the meridional plane, and $\sigma_3 = \sigma_\theta$ is the circumferential stress.

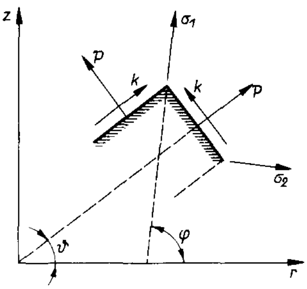

FIG. 1. Slip-line convention.

Since equations (1) and (2) impose two independent conditions on the four stress components $\sigma_r \sigma_z \tau_{rz}$ and σ_θ, the state of stress may be expressed by two independent parameters ϑ and p, whose meaning is indicated in Fig. 1. Consequently we can write

$$\left.\begin{aligned} \sigma_z &= p + k\sin 2\vartheta, & \tau_{rz} &= k\cos 2\vartheta, \\ \sigma_r &= p - k\sin 2\vartheta, & \sigma_\theta &= p - k. \end{aligned}\right\} \tag{3}$$

The substitution of the expressions (3) into the equilibrium equations yields a hyperbolic system of quasi-linear partial differential equations with two unknown functions p, ϑ and two independent variables r, z. The equations of the characteristics of this system have the form

$$\frac{\mathrm{d}z}{\mathrm{d}r} = \tan\vartheta, \qquad \mathrm{d}p - 2k\,\mathrm{d}\vartheta = \frac{k}{r}(\mathrm{d}z - \mathrm{d}r) \tag{4a}$$

for the first family of lines, called later the α-family, and

$$\frac{\mathrm{d}z}{\mathrm{d}r} = -\cot\vartheta, \qquad \mathrm{d}p + 2k\,\mathrm{d}\vartheta = -\frac{k}{r}(\mathrm{d}z + \mathrm{d}r) \tag{4b}$$

for the second family, called the β-family.

The vector of the flow velocity in the meridional plane r, z can be expressed by the components v_α and v_β along the α and β lines, respectively. If use is made of the isotropy condition and of the incompressibility condition, these components must satisfy the equations

$$\left.\begin{aligned} \mathrm{d}v_\alpha - v_\beta\,\mathrm{d}\vartheta &= -(v_\alpha \cot\vartheta - v_\beta)\,\mathrm{d}z/2r \qquad \text{along an } \alpha\text{-line} \\ \mathrm{d}v_\beta + v_\alpha\,\mathrm{d}\vartheta &= (v_\alpha \cot\vartheta - v_\beta)\,\mathrm{d}r/2r \qquad \text{along a } \beta\text{-line.} \end{aligned}\right\} \tag{5}$$

Equations (5) were given by Hill [3] and later by Shield [1].

The solution of particular problems consists in numerical integration of the equations (4) and (5) by means of the well known Masseau method. Solving subsequently appropriate boundary value problems for stresses and velocities, we finally obtain the stress and velocity field in the plastic region of the bar.

V-NOTCHED BARS

Let us consider an axially symmetric bar with a V-shaped notch with the total angle 2γ (Fig. 2). The bar is loaded by two opposite tensile forces. Numerical solutions were obtained for notches with angles $2\gamma = 60°$ and $120°$. For the slit-notch ($2\gamma = 0°$) all necessary values were taken from Shield's solution [1] for the punch indentation problem. The procedure of solution is described for the 120°-notch.

The yield point load was obtained from the slip-line field shown in Fig. 3. On the free edge AB we have $\sigma_1 = 2k$, $\sigma_2 = \sigma_3 = 0$, and therefore $p = k$ and $\vartheta = \gamma - (\pi/4)$.

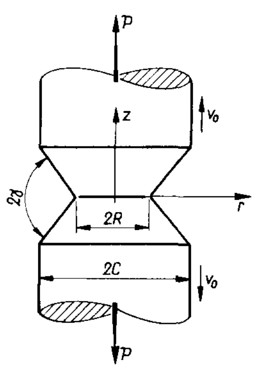

FIG. 2. V-notched bar.

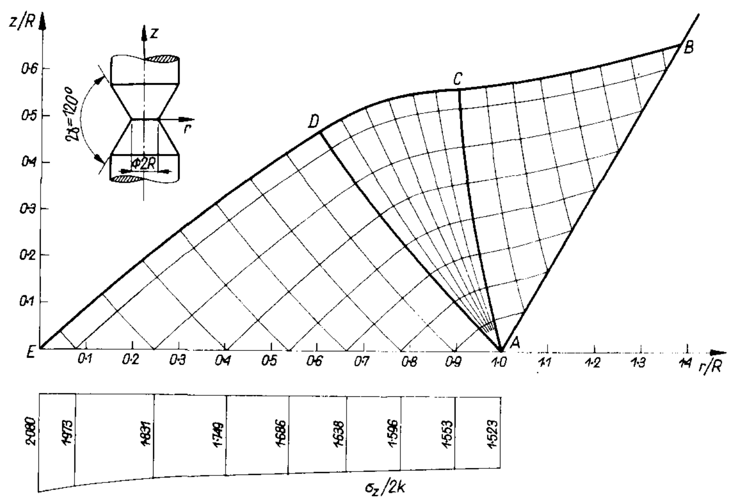

FIG. 3. Slip-line field for $2\gamma = 120°$.

Thus, solving the Cauchy boundary value problem the field of characteristics in the region ABC can be found. The position of the point B is not known initially and will be determined later by the condition that the α-line starting from B must pass through E. The β-line AC, along which the values of p and ϑ are now known, and the known value $\vartheta = \pi/4$ along AE, constitute the mixed boundary value problem and therefore define the field ACDE. The point A is a singular point since the value of ϑ at AB is different from the value at A on AE. The angle of the fan CAD at A is equal to $\pi/2-\gamma$.

Having found the values of p along AE, we can obtain σ_z from the first expression (3). The distribution of σ_z along AE is shown in Fig. 3. The total load of the incipient plastic flow can be obtained by means of numerical integration from the following relation

$$P^* = 2\pi \int_0^R \sigma_z r \, dr. \tag{6}$$

The yield point load factor of the notched bar will be defined as the ratio

$$f = P^*/P_0, \tag{7}$$

where $P_0 = 2\pi R^2 k$ is the yield point load of the smooth bar with the diameter $2R$.

Figure 4 shows the slip-line field for the notch angle $2\gamma = 60°$ and the distribution of the stress σ_z along the radius in the minimal cross section of the bar.

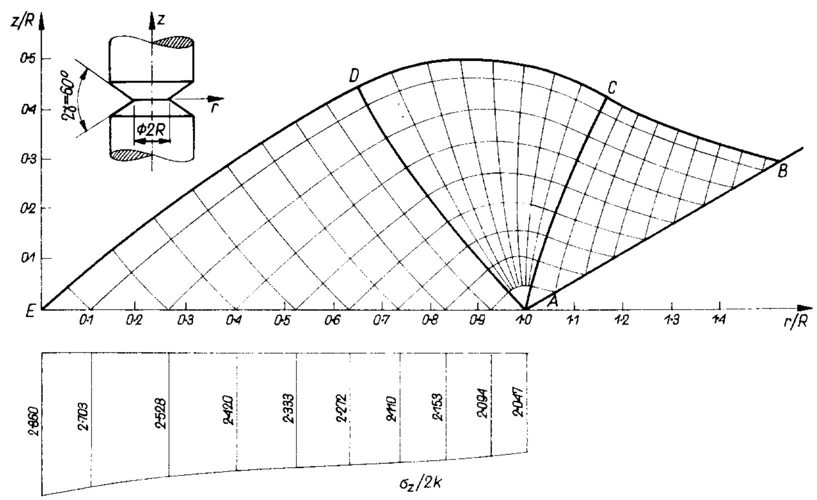

FIG. 4. Slip-line field for $2\gamma = 60°$.

The obtained load factors for $2\gamma = 120°$ and $60°$ are marked in Fig. 5. Their values are $f_{60°} = 2{\cdot}32$ for $2\gamma = 60°$ and $f_{120°} = 1{\cdot}65$ for $2\gamma = 120°$. For slit notch ($2\gamma = 0°$) the value of the factor $f_{0°} = 2{\cdot}85$ is taken from Shield's solution [1] for the punch indentation problem. For $2\gamma = 180°$ the load factor is, of course, equal to unity. The line plotted through the calculated points gives the value of the load factor for an arbitrary angle of the notch. The other line shows the load factor values for plane strain bars with V-shaped notches.

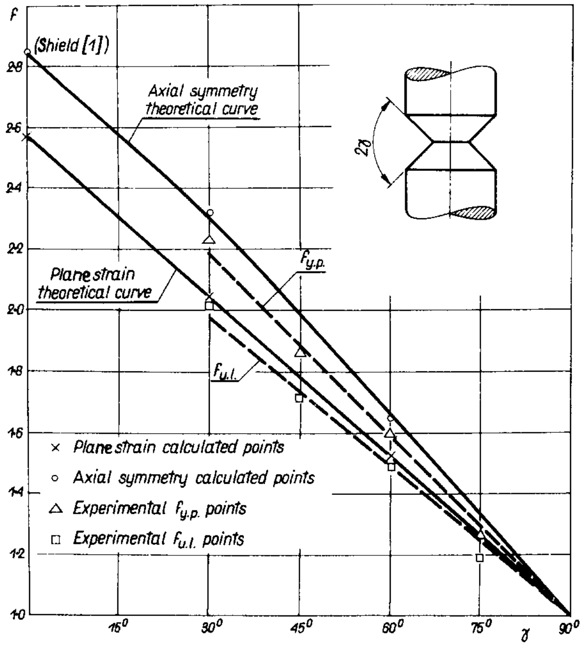

FIG. 5. Theoretical and experimental values of the load factors.

Finally let us consider the velocity field compatible with the field of characteristics ABCDE and satisfying the velocity boundary conditions. The rigid parts of the bar move with the velocity v_0 as shown in Fig. 2. However, it is more convenient to assume, without loss of generality, that the upper part of the bar is fixed ($v = 0$) and the lower part moves downwards with velocity $2v_0$. Such conditions can be obtained by superposition of the rigid translation of the bar downwards with the velocity v_0.

Since it is assumed that the α-line BCDE separates the regions of plastically deforming and non-deforming material the normal velocity v_β across this line must be zero. Thus along BCDE $v_\beta = 0$ and the first of equations (5) gives

$$v_\alpha = A/\sqrt{r} \quad \text{on BCDE},$$

where A is a constant. To avoid an infinite value of v_α at E, the constant A must be set equal to zero. Thus

$$v_\alpha = v_\beta = 0 \quad \text{on BCDE}. \tag{8}$$

The second condition for velocities is that $v_z = -v_0$ or

$$v_\alpha + v_\beta = -(\sqrt{2})v_0 \quad \text{along AE}. \tag{9}$$

The conditions (8) and (9) constitute the mixed boundary value problem for the velocity equations (5), and define the velocity field in ABCDE. Since the velocity problem is similar to that considered by Shield [1], the same procedure may be used in our case to determine the velocity field. In the present study the velocity field has not been calculated, and therefore the numerical check of the evident inequalities

$$\dot{\varepsilon}_1 \geq 0, \qquad \dot{\varepsilon}_2 \leq 0, \qquad \dot{\varepsilon}_3 \leq 0$$

could not be made.

McClintock [2] pointed out that the slip-line field, extended into the non-deforming region given by Shield, can also be applied to the case of the bar with a slit-shaped notch. It is evident that for an arbitrary angle γ of the notch there exists the stress field extended into the rigid part of the bar. Thus the solutions presented above are statically admissible, provided the diameter outside the notch is not smaller than the largest diameter of the extended stress field.

Thus the theoretical value of the diameter $2C$ can be obtained only on the basis of the extended slip-line field. The amount of work connected with this procedure demands the use of digital computers.

However, for practical purposes, the diameter $2C$ can be evaluated in a simple manner from the analogous solutions for the plane-strain notched bars. As shown by Bishop [4], in plane strain conditions the extended slip-line field can be easily obtained graphically, giving the required H/h ratio (Fig. 6). Let us assume that the required ratio of the cross

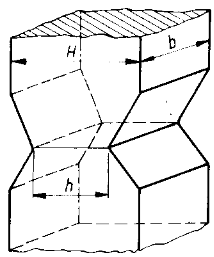

FIG. 6. Plane strain notched bar.

section areas F_{max}/F_{min} is the same for axially symmetric and plane strain bars with the same parameters of the notch. For axially symmetric bar $F_{max} = \pi C^2$, $F_{min} = \pi R^2$, and for plane strain bar $F_{max} = bH$, $F_{min} = bh$. Therefore, we obtain the relation

$$C/R = \sqrt{(H/h)}, \tag{10}$$

from which the required ratio C/R can be estimated on the basis of the H/h ratio for plane strain bar with the same notch.

For example, for a plane strain bar with a slit-notch Bishop [4] found $H/h = 8{\cdot}67$, whereas for an axially symmetric bar the extended stress field [1] gives $C/R = 3{\cdot}2$. Formula (10) gives the value $C/R = \sqrt{8{\cdot}67} = 2{\cdot}94$, which is 8·1 per cent smaller than the exact value.

Figure 7 shows the value of the C/R ratio for V-notched axially symmetric bars with various angles γ. The diagram was obtained by means of the relation (10) from the analogous diagram for the H/h ratio given by McClintock [2] for plane strain bars.

Experiments were performed to determine the yield point and ultimate load for V-notched axially symmetric bars with various angles γ. The extension of the plastically

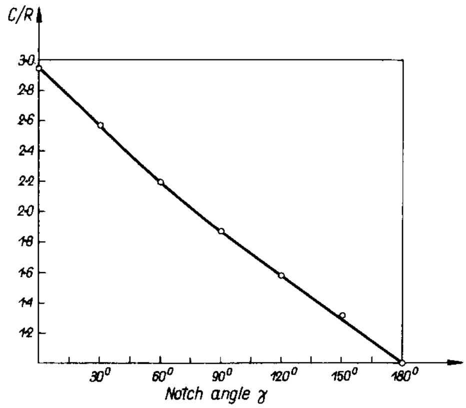

FIG. 7. Theoretical values of the C/R ratio for V-notched bars.

deformed part of the free surface was also investigated. The specimens had the dimensions $R = 15$ mm and $C = 30$ mm. The angle γ had the values 75°, 60°, 45° and 30°. Moreover, a specimen with no notch $\gamma = 90°$ was prepared with the radius $R = 15$ mm. The material was technically pure aluminium (99·7% Al).

A universal hydraulic testing machine and hinge-type fittings were used in order to avoid the possible bending of the bar. Deformations were recorded by means of an Amsler mechanical extensometer with a 0·01 mm division dial gauge and 60 mm gauge length.

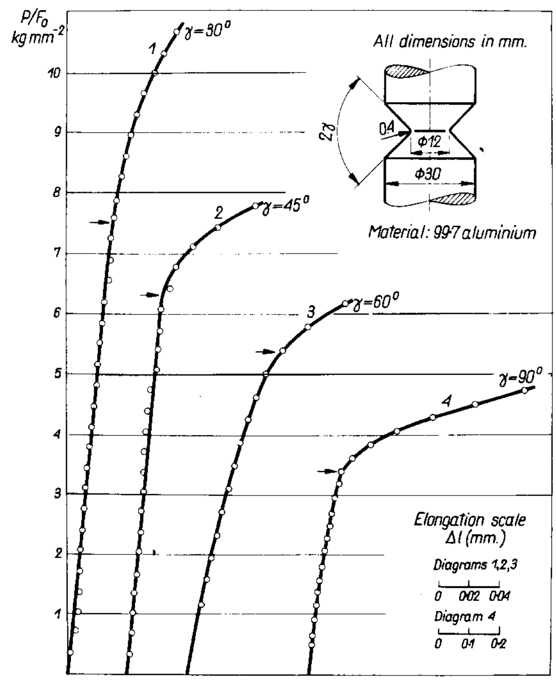

FIG. 8. Initial parts of the stress–strain diagrams.

Figure 8 shows the initial parts of the obtained stress–strain diagrams. Arrows indicate the estimated average yield point stresses in the minimum section of the bar. The actual yield point load factor for a given γ is equal to $f_{\text{y.p.}} = Y_\gamma/Y^\circ$, where Y_γ is the yield stress for the notched bar and Y° the yield stress for the smooth bar ($\gamma = 90^\circ$): In the same manner the actual ultimate load factor can be obtained. If R_γ is the maximum nominal stress for the notched bar, and R° is the maximum nominal stress for the smooth bar ($\gamma = 90^\circ$), the ultimate load factor is $f_{\text{u.l.}} = R_\gamma/R^\circ$. Both actual load factors obtained from experimental investigation are shown in Fig. 5.

The theoretical velocity solution presented above indicates that the plastically deforming region is bounded by the slip-lines BCDE shown in Figs. 3 and 4. The actual deforming region is, however, larger as shown in Fig. 9, where the upper rigid part of the specimen is assumed to be immovable. This effect can be attributed to the strain hardening of the material, and assumed simplifications in the theoretical solution.

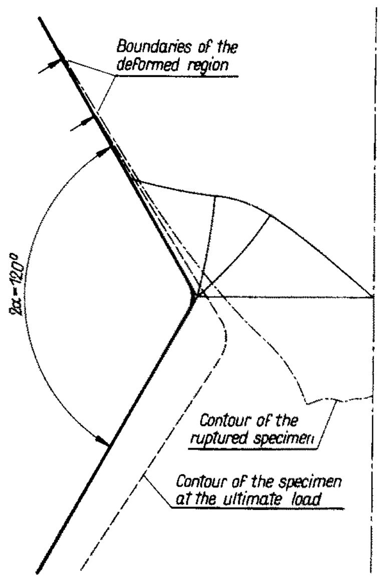

FIG. 9. Deformation mode of a V-notched bar.

BARS WITH ROUNDED NOTCHES

In this section notches are investigated, with the cylindrical central part bounded by two rounded parts (Fig. 10). Theoretical solutions are obtained for a circular notch with $a = \rho_0$ and for a notch with $a = 0{\cdot}8R$. The ratio $\rho_0/R = 0{\cdot}2$ is constant for all notches.

Figure 11 shows the slip-line field for the circular notch ($a = \rho_0$) and the distribution of the axial stress σ_z in the minimum cross section OB. The method of solution is similar to that presented above for the V-shaped notch, except that there is no singularity in B.

In the slip-line field for a rounded notch ($a = 0{\cdot}8R$), shown in Fig. 12, the triangle ABC represents the region of uniaxial tension with $\sigma_z = 2k$, $\sigma_r = \sigma_\theta = 0$. The boundary conditions on BD ($p = k$, $\vartheta = \pi/4-\gamma$) define the stress field BED. Obtained values of p and ϑ along BE and given values along BC ($p = k$, $\vartheta = \pi/4$) constitute the characteristic problem for the field BCFE. Finally, in the curvilinear triangle COF we have the mixed boundary value problem, since $\vartheta = \pi/4$ along the non-characteristic line CO.

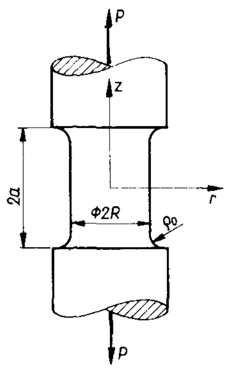

FIG. 10. Bar with rounded notch.

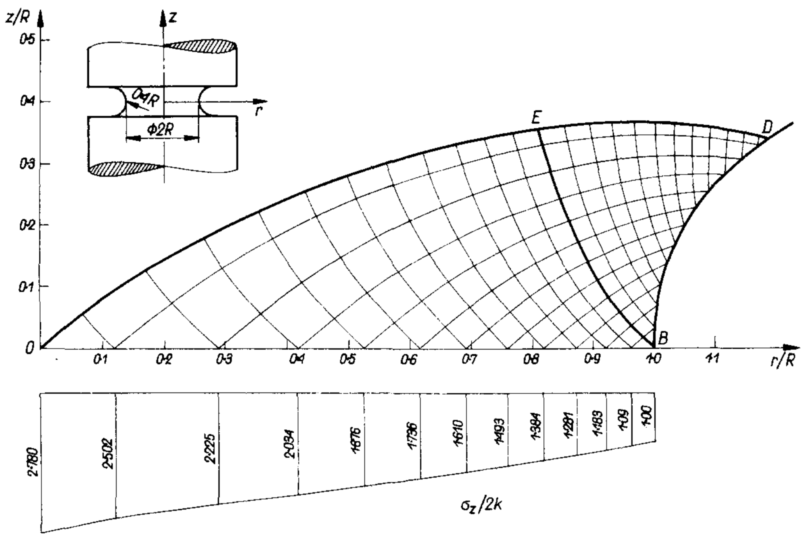

FIG. 11. Slip-line field for circular notch.

The velocity solution can be obtained in the same manner as for the V-notch.

The yield point load factor is given by (7), where $P_0 = 2\pi R^2 k$ is the yield load for the bars with $(a-\rho_0) \geq R$. The theoretical values of the factor for various a/R are represented by lower line in Fig. 13. The upper line shows the yield point load factor for plane strain bars with analogous notches. It is interesting to note that plane strain factors are now larger than factors for axial symmetry, whereas they were smaller for V-shaped notches (Fig. 5).

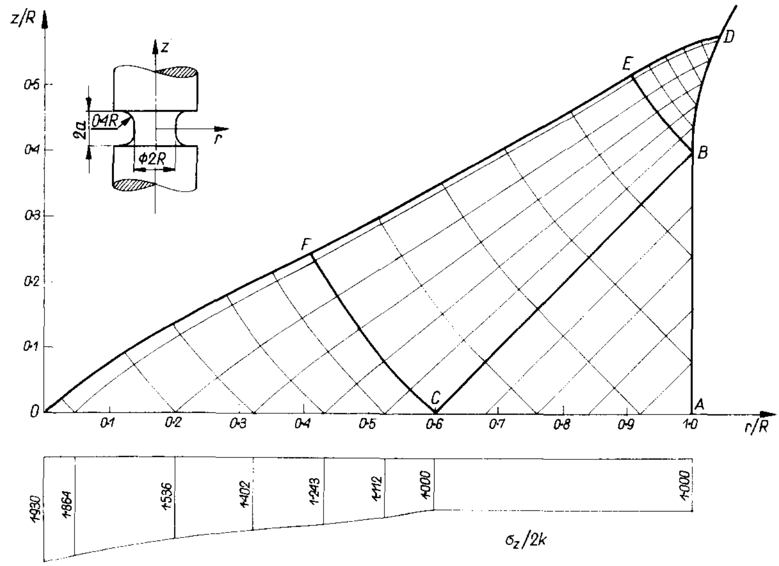

FIG. 12. Slip-line field for rounded notch.

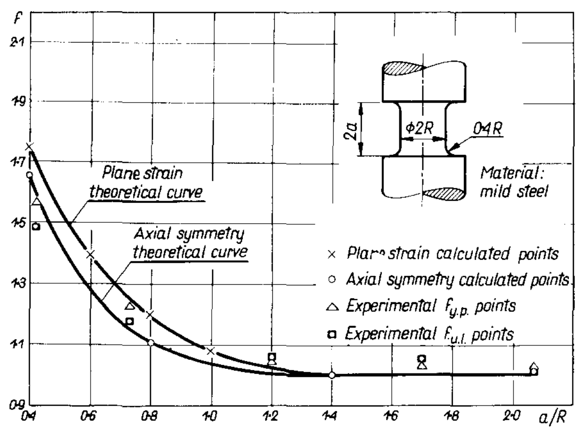

FIG. 13. Theoretical and experimental values of the load factors.

The actual yield point and ultimate load factors were investigated using mild steel (0·15%C) specimens with $2R = 5\,\text{mm}$, $\rho_0 = 1\,\text{mm}$ and various a/R ratios shown in Fig. 14. This figure shows the initial parts of the stress–strain diagrams, from which the average yield point stresses Y were estimated. Both actual factors $f_{y.p}$ and $f_{u.l.}$ were obtained in the same manner as for V-notched specimens. The experimental points are shown in Fig. 13.

The last set of specimens was used to investigate the influence of the C/R ratio on the yield point and ultimate load of the bars with the circular notch ($a = \rho_0$). The radii $\rho_0 = 1\,\text{mm}$ and $2R = 5\,\text{mm}$, and the material were the same as in the previous test.

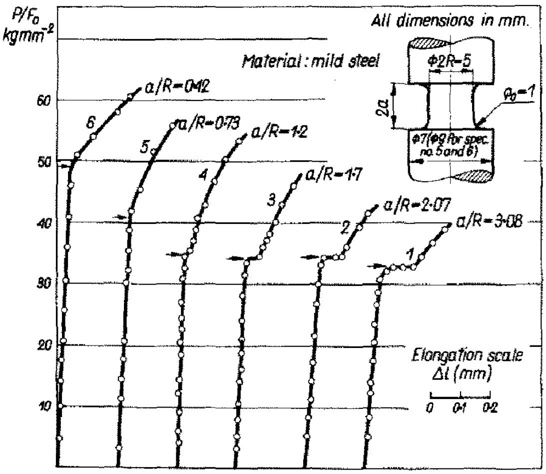

FIG. 14. Initial parts of the stress–strain diagrams.

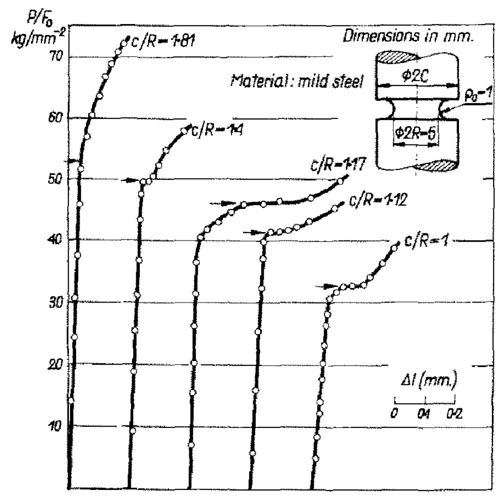

FIG. 15. Initial parts of the stress–strain diagrams.

The chosen values of the C/R are shown in Fig. 15. The yield loci were estimated from the initial parts of the stress–strain diagrams presented in Fig. 15. The experimental points for yield locus average stress $\sigma_{y.p.}$ and ultimate nominal stress $\sigma_{u.l.}$ are marked in Fig. 16. The theoretical value of the C/R ratio was found from the relation (10) to be

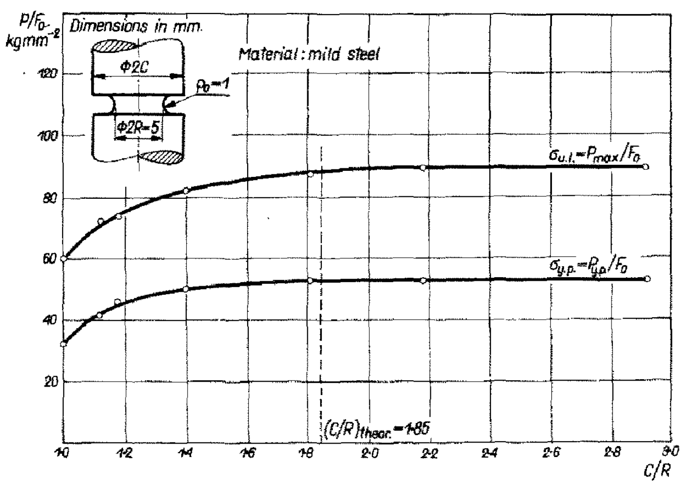

FIG. 16. Influence of the C/R ratio on the load factors.

$C/R_{\text{theor}} = 1{\cdot}85$. The ratio H/h for the analogous plane strain problem was obtained graphically, using the method proposed by Bishop [4].

FINAL REMARKS

For practical purposes an important feature of the presented axially symmetric solutions is that they differ only slightly from those for plane strain conditions. The largest difference is 9·8 per cent for the slit-notch. The load factors obtained for axially symmetric bars can be larger or smaller than the load factors for plane strain bars, depending on the shape of the notch.

The agreement between the theory and the experimental results is satisfactory. It is important to note that for materials used in the present experiments, the theoretical load factors obtained for perfectly plastic material, are also valid for the ultimate load of the notched bars.

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Résumé—Des résultats théoriques et expérimentaux sont présentés pour l'écoulement plastique d'une barre entaillée axiellement symmétrique. Les champs de ligne de glissement ont été numériquement calculés employant la théorie de Shield de l'écoulement plastique axiellement symmétrique. Les facteurs de chargement obtenus des solutions théoriques ne diffèrent que légèrement de ceux des conditions de contrainte plane. Deux séries de spécimens ont été investiguées afin d'obtenir les facteurs de chargement pour des entailles en forme de V et arrondies. Les similitudes entre la théorie et les résultats expérimentaux sont satisfaisantes. Dans la dernière série d'experiments l'influence du diamètre de la barre hors de l'entaille a été investiguée et comparée, plus tard, à la valeur estimée théoriquement de ce diamètre.

Zusammenfassung—Theoretische und Versuchs Ergebnisse sind für das plastische Fliessen von achsensymmetrischen gekerbten Stäben gegeben. Die Gleitungslinienfelder wurden berechnet bei Verwendung der Schildtheorie für achsensymmetrisches plastisches Fliessen. Die Belastungsfaktoren, erhalten von den theoretischen Lösungen, sind nur geringfügig unterschiedlich von denen für Flächenbeanspruchungs Bedingungen. Zwei Probensätze wurden untersucht, um die tatsächlichen Belastungsfakoren für V-geformte und abgerundete Kerbe zu erhalten. Die Übereinstimmung zwischen den Theoretischen und Versuchs Ergebn ssen ist zufriedenstelland. In der letzten Serie der Versuche, der Einfluss des Durchmessers des Stabes ausserhalb der Kerbe wurde untersucht und später mit den theoretisch geschätzten Werten dieses Durchmessers verglichen.

Абстракт—Даются теоретические и экспериментальные результаты для пластического течения симметрических относительно оси надрезанных брусков (стержней). Поля линии скольжения вычисляются применением теории осесимметрического пластического течения Шильда. Факторы нагрузки, полученные теоретическими решениями только слегка отличаются от факторов для условий плоской деформации. Для того, чтобы получить действительные факторы нагрузки для V-образных и кругоьых нареȝов исследовалось два набора образцов. Согласованность между теорией и зкспериментальными результатами—удовлетворительна. В последних сериях зкспериментов исследовалось влияние диаметра бруска снаружи надреза и позже оно сравнивалось с теоретически подсчитанным вычислением зтого диаметра.